

# "Gauge" freedom and relationship between the Einstein and Jordan conformal frames

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## Abstract

The issue of the physical equivalence between the Einstein and Jordan conformal frames in Jordan-Brans-Dicke (JBD) theory is revised. Scalar-tensor theories equations are not invariant with respect to conformal transformations if one uses the same "gauge" fixing in Jordan and Einstein frames. We hope to have clarified some eventual obscure issues associated to the concept of conformal invariance of relativistic theories appearing in the literature, in particular the relationship between Jordan and Einstein frames.

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## I. INTRODUCTION

One of the largest concentrations of literature within the area of relativistic gravity theories is interpretations of exact solutions of field equations. Some of them are discovered at the early stage of development of relativistic theories, but up to now they are often considered as equivalent representations of some "unique" solution. On the other hand, the variables  $x$ , used in Einstein's equations, represent co-ordinates of points of abstract four-dimensional manifold  $M^4$  over which there are a set pseudo Rimanien spaces  $V^4(g)$ , generated by set of solutions  $g_{\alpha\beta}$ . Co-ordinates in each of such spaces have the specific properties differing from their properties in other spaces [1]. Note that an affine connection which to attach to gravity theory can be at most an independent postulat of theory [2]. In this case the point dependent property of manifolds is linked with the fact that the units for measure of underlying geometry are running units. For example, it could be a theory based on Brans Dicke action but endowed with space time of Weyl integrable structure [3].

Evidently that needs preliminary a clear mathematical distinction between the concepts of co-ordinate, "gauge", reference frames in the relativistic theories. To obtain the unknown components of metric tensor from the field equations we must to do the following conventions:

First, sets of numbers, with whom we conduct manifold arithmetization (e.g. spherically symmetric). Before solving field equations we have only a differential manifold structure endowed with an affine connection. The field equations of relativistic theory can be derived from the action up to boundary terms. The angles and distance are induced rather than fundamental concepts in this proposal. It is important to note that the two geometrical structures, the metric and arithmetization, are fundamentally independent geometrical objects.

Second, from geometrical point of view one has to introduce an additional mathematical structure - describing some specific principle of construction of space-time model is responsible for measuring the distances - the "gauge". On the other words "gauge" is a rule for reception of "coordinate system" on a single manifold  $M$  (e.g. harmonic, isotropic, curvature coordinates). It is important to realize that the field equations alone are not enough to determine a gravitational system, while these equations are a set of 6 nonlinear partial differential equations for the 10 metric components. Einsteins equations determine the solution of a given physical problem up to four arbitrary functions, i.e., up to a choice of "gauge".

Evidently, a structure of space-times is mathematically represented by Einsteins equations and four co-ordinate conditions [4], which considered independent of the action

$$G_{\mu\nu} = T_{\mu\nu}, \quad (1a)$$

$$C(x)g_{\mu\nu} = 0, \quad (1b)$$

where  $g_{\mu\nu}$  metric tensor and  $C(x)$  - some algebraic or differential operators. Thereby for any four of components  $g_{\mu\nu}$  emerge the relations with remaining six and, probably, any others, known functions. Certainly, equations (1b) cannot be covariant for the arbitrary transformations of independent variables, and similarly should not contradict Einsteins equations or to be their consequence. Moreover, four of ten field equations will not be transformed according to any rules, but simply replaced by hand with the new. This "gauge" is the unphysical degree of freedom and we must fix the "gauge" or extract some invariant quantities to obtain physical results [5].

Third, the unknown components of metric tensor  $g_{\mu\nu}$  are determined from the solutions of Einstein's field equations. (e.g. Schwarzschild, Heckmann, Brans solutions). Consequently, the geometrically interpreted co-ordinate system of obtained space-time and any relationship it derives from equations (1a), (1b) emerge a posteriori [4]. Moreover, property of this co-ordinate system will depend from initial and boundary conditions for (1a), (1b). An intriguing consequences of the above discussion is the "gauge" freedom can be expected in relation with some connection to problems in quantum physics. Generally speaking, occurrence of the observer ("gauge" fixing) influences results of measurements and physics are different in two different "gauges".

## II. CONFORMAL TRANSFORMATION (WEYL RESCALING)

The important feature of the JBD gravity is connected with the conformal symmetry. It is well known, since the pioneering paper of Jordan [6] that the action of a scalar tensor theory is invariant under local transformations of units that are under general conformal transformations, or sometimes called Weyl rescaling:

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \text{or} \quad ds^2 \rightarrow d\hat{s}^2 = \Omega^2(x)ds^2. \quad (2)$$

where  $\Omega(x)$  a local arbitrary function of  $x$ .

This method of conformal transformation provides a clear and powerful technique, free from mathematical ambiguity, but nevertheless requires careful consideration from the physical point of view.

Among all conformally related frames one distinguishes two frames: Jordan's and Einstein's. Let us consider the pure gravitational sector of the JBD theory (as a minimal extension of general relativity) in the Jordan's conformal frame, the field equations (1a) can be derived from the following action

$$L(g, \phi) = \sqrt{-g} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + L_{matter}(g) \quad (3)$$

where  $R$  is the curvature scalar,  $\phi$  is the scalar JBD field,  $\omega$  is the JBD coupling constant, and  $L_{matter}[g]$  is the Lagrangian density of the ordinary matter minimally coupled to the scalar JBD field.

The gravitational part of the Jordan's frame JBD Lagrangian density  $L(g, \phi)$  (3) without ordinary matter is invariant in form under the conformal rescaling of the spacetime metric [6], [7]:

$$g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \phi^{2\alpha} g_{\mu\nu} \quad (4)$$

After the conformal transformations (4), we get Lagrangian for the Einstein's frames

$$L(\hat{g}, \hat{\phi}) = \sqrt{-\hat{g}} \left( \hat{R} - \left( \omega + \frac{3}{2} \right) \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\phi} \hat{\nabla}_\nu \hat{\phi} \right) + \hat{L}_{matter}(\hat{g}, \hat{\phi}) \quad (5)$$

The scalar function  $\hat{\phi} \equiv \ln \phi$  is the JBD scalar field in the Einstein frame, and  $\hat{L}_{matter}(\hat{g}, \hat{\phi})$  is the Lagrangian density for the ordinary matter nonminimally coupled to the scalar field.

Note the possibility of changing the coupling in (3);  $L_{matter}(g) \rightarrow L_{matter}(g, \phi)$ , while keeping intact the gravitational part:

$$L^*(g, \phi) = \sqrt{-g} \left( \phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + L_{matter}(g, \phi) \quad (6)$$

In this case the ordinary matter is nonminimally coupled to the scalar field in the Jordan frame.

In addition one can obtain the Lagrangian density to the Einstein frame in the form:

$$L^*(\hat{g}, \hat{\phi}) = \sqrt{-\hat{g}} \left( \hat{R} - \left( \omega + \frac{3}{2} \right) \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\phi} \hat{\nabla}_\nu \hat{\phi} \right) + \hat{L}_{matter}(\hat{g}) \quad (7)$$

In this case the scalar field  $\hat{\phi}$  is minimally coupled to the curvature. Note that, unless a clear statement of what is understood by "equivalence of frames" - is made, the issue which is the physical conformal frame is a semantic one. Hence, there are four related but inequivalent scalar-tensor theories in Jordan and Einstein frame [11].

In the literature, the physicists do not agree with each other about the equivalence of the two frames (see review in [8]). However, the meaning of the equivalence between the Jordan frame and the Einstein frame is not assuming the additional equations (1b). These equations put by hand and not covariant. This issue is critical for the interpretation of the predictions of a given theory of gravity since these seem to be deeply affected by the choice of the coordinate conditions [1]. For concreteness, let us consider "harmonic gauge" of coordinate [9],

$$g_{\mu\nu}\Gamma^\alpha_{,\mu\nu} = 0. \quad (8)$$

which usually assumed as the analogue of Lorenz gauge,  $\partial A = 0$ , in electromagnetism. However this analogy is the most superficial: this or that gauge in nonrelativistic theory is a problem of exclusively convenience, it's this or that expedient does not influence in any way on a values of physical quantities and it is not related to observation requirements, - whereas the choice of co-ordinate system is related to all it essentially.

In fact, there are the related but inequivalent scalar-tensor theories in Jordan and Einstein frame. The reason is very simple. If we use the same conformal transformations, like the (4), in both the equations (1a) and (1b), then the in and out states are not the same in the two frames. If one postulates that the field equations are invariant with respect to conformal transformations (1b), one obtains in addition transformations of co-ordinate conditions (9)

$$g_{\mu\nu}\Gamma^\alpha_{,\mu\nu} = \hat{g}_{\mu\nu}\hat{\Gamma}^\alpha_{,\mu\nu} + \frac{\partial_\mu\Omega}{\Omega}. \quad (9)$$

As a result, since the Einstein field equations are undetermined; scalar-tensor theories cannot achieve the harmonic metric for any  $\Omega$  functions but only when  $\Omega$  is taken a constant. One must assume that two frames represent not the same set of physical gravitational and non-gravitational fields. In fact, two conformally connecting spaces  $V^4(g)$  and  $\hat{V}^4(g)$  are given not in the same manifold. Consequently, under this conformal transformation the solution

of some initial physical problem will be transformed onto a solution of a completely different problem. Thus, applying the same coordinate conditions in different physical requirements, we arrive at dissimilar physical theories, because we are solving different equations.

On the other hand each scalar-tensor theory can be transformed into general relativity plus conformally invariant scalar field [10]. The gravitational interaction for scalar tensor theories is taken into account by the Einstein equations, which are generally written in the form (1a). The Einstein tensor  $G_{\mu\nu}$  is constructed from the geometrical properties of the space-time, while  $T_{\mu\nu}$  is the energy momentum tensor of matter. One can in principle assume gauge-dependence of right-hand-side of equation (1a) as a variety of matter fields with different equations of state. Now, if we consider, the system (1a), (1b) as equations for same "gauge" fixing then the Jordan's and Einstein's conformal frames can be viewed as a different "matter source" of energy momentum tensors  $T_{\mu\nu}$  of Einstein's equations. The physical content of this point of view can be stated in the following simple way: the equations of state for "matter" are not the same for the different conformal frames is chosen.

It is evident that in different conformal frame representations of JBD theory are neither mathematically, nor physically equivalent.

### III. CONCLUSION

In this article, we clarify the notion of "gauges" in relativistic theories, which is necessary to understanding the physical equivalence between the Einstein and Jordan conformal frames. We have shown the procedure to find gauge-invariant variables in the scalar tensor theories through the precise treatments of "gauges". In method of conformal transformation, we always treat two space-time manifolds. One is the space-time for Jordan frame and the other is the space-time for Einstein frame. Note that these two space-times for Jordan and Einstein frame are distinct. In any relativistic theories, we must always write additional four co-ordinate conditions (1b). These "gauges" may describe different physical solutions of Einstein equations with the same space arithmetization. The conformal transformations are not diffeomorphisms of the single manifold  $M$ , and the transformed metric  $\hat{g}_{\mu\nu}$  is not simply the metric  $g_{\mu\nu}$  written in a different coordinate system these metrics describe different gravitational fields and different physics.

Keeping in our mind that we always treat two different space-times in Jordan and Einstein

frame. Eq. (1b) is a rather curious equation because it not covariant for the arbitrary transformations of independent variables. In this case the metric is left unchanged, although its coordinate representation varies. In short, Eq. (1b) gives a relation between variables on two different manifolds.

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- [1] A. Gullstrand, Allgemeine Lösung des statischen Einkörperproblem in der Einsteinschen Gravitationstheorie. Ark. for Mat., Astr. o. fysik, Bd.16, N 8, (1921)
  - [2] C. H. Brans, Mach's Principle and a Relativistic Theory of Gravitation. II Phys. Rev. **125**, 2194 (1962).
  - [3] I. Quiros, R. Garcia-Salcedo, J. E. M. Aguilar and T. Matos, The conformal transformation's controversy: what are we missing? , arXiv:1108.5857.
  - [4] A.N.Temchin, Uravneniia Einshteina Na Mnogoobrazii, Moskow, URSS,1999, (Russian).
  - [5] Kouji Nakamura, "Gauge" in General Relativity: Second-order general relativistic gauge-invariant perturbation theory, Bulg. J. Phys. vol.35 no.s1 (2008), pp. 489-492
  - [6] P. Jordan, Schwerkraft und Weltall, Vieweg (Braunschweig) 1955.
  - [7] V. Faraoni, Phys. Lett. A **245**, 26(1998).
  - [8] Y. Fujii, N. Maeda, The scalar-tensor theory of gravitation (Cambridge, Cambridge University Press, 2003).
  - [9] V. A. Fock, "The Theory of Space, Time and Gravitation". Macmillan. (1964).
  - [10] L. M. Sokolowski Universality of Einstein's General Relativity, GR14 Conference (Florence, Italy, Aug 1995) (1995).
  - [11] I. Quiros, "Conformal classes of Brans-Dicke gravity", arXiv:gr-qc/9904004v2.